Splitting tournaments*

Edwin Leuven[†] Hessel Oosterbeek[‡] Bas van der Klaauw[§]

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Abstract

In this paper we investigate how heterogeneous agents choose among tournaments with different prizes. We show that if the number of agents is sufficiently small, multiple equilibria can arise. Depending on how the prize money is split over the tournaments, these may include, for example, a perfect-sorting equilibrium in which high-ability agents compete in the high-prize tournament, while low-ability agents compete for the low prize. However, there are also equilibria in which agents follow a mixed strategy and there can be reverse sorting, i.e. low-ability agents are in the tournament with the high prize, while high-ability agents are in the low-prize tournament. We show that total effort always decreases compared to a single tournament. However, splitting the tournament may increase the effort of low-ability agents.

Keywords: self-selection, tournament, heterogeneous agents, social planner.

1 Introduction

We start by introducing four motivating examples:

1. Instead of one single contest over 42.195 kilometers, the organizers of the Leyden marathon organize runs for the classical distance and for the half distance, the 10k and the 5k. The winner of the whole marathon receives a prize of 1000 Euros, and the winners of the half marathon, the 10k and the 5k receive prizes of 500, 500 and 250 Euros, respectively. Many races have a

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[†]CREST (ENSAE). Also affiliated with the CEPR, CESifo and IZA. edwin.leuven@ensae.fr

[‡]University of Amsterdam and Tinbergen Institute. h.oosterbeek@uva.nl

[§]VU University Amsterdam, Tinbergen Institute, CEPR and IZA. bklaauw@feweb.vu.nl

similar setup. For example, the Amsterdam marathon pays prize money to the winner of the marathon, but only an award to the winner of the half distance. Similar, to qualify for the prestigious Ironman triathlon in Hawaii, people can register at a qualification race as professional or as a so-called age-grouper.

2. For the recruitment of junior researchers, many universities do not only open vacancies for tenure track positions but also for postdoctoral researchers. Those who are hired for a tenure track position win a different “prize” than those hired as postdoctoral researchers.

3. A young talented European economist can move to the US to try to win the Bates Clark medal for best economist in the US under 40, or can move to Australia to try to win the Young Economist Award from the Economic Society of Australia.[1]

4. A gifted swimmer with an arms span of 201 cm, relatively short legs, feet size 14 and hypermobile ankles can try to win 8 gold Olympic medals for 200m freestyle, 100m and 200m butterfly, 200m and 400m individual medley, 4×100m and 4×200m freestyle relay and 4×100m medley relay, or can try to win the most prestigious gold Olympic medal for the 100m freestyle.

In each of these examples agents have to decide in which tournament they want to compete, where in the first and third example participation in multiple tournaments is impossible. In the first, second and fourth example, a single organizer has split the total prize money over multiple tournaments.

Since Lazear and Rosen (1981) it has been recognized that tournaments can be very useful in providing incentives. Lazear and Rosen, but also Green and Stokey (1983) and Nalebuff and Stiglitz (1983) analyze internal labor markets as tournaments. An important theoretical result is that rank-order tournaments can yield optimal level of effort while, in contrast to piece rates, only relative instead of absolute performance has to be observed to provide correct incentives.[2]

More recent contributions focus on the optimal prize structure of tournaments (Glazer and Hassin, 1988; Moldovanu and Sela, 2001; see also Clark and Riis, 1998) or on the optimal structure of tournaments (Moldovanu and Sela, 2006). Moldovanu and Sela (2001) analyze the question how a fixed amount of prize money should be

[1]We thank Paul Frijters (the 2009 winner of the Young Economist Award from the Economic Society of Australia) for pointing us to this example.

[2]The empirical evidence in support of this model comes mainly from laboratory experiments (Bull et al., 1987; Harbring and Irlenbusch, 2003; Schotter and Weigelt, 1992) and from sports tournaments (Ehrenberg and Bognanno, 1990; Sunde, 2009). Leuven et al. (2008) argue that due to sorting the external validity of this evidence is limited.

divided between several possibly different prizes in a single tournament (a first prize, a second prize etc.). In their model the agent choosing the highest effort wins the first prize and agents have private information about their ability (cost of effort). They show that if the cost function of effort is linear or concave then total effort is maximized when there is only one large prize. It might be optimal to split the prize money in more than one prize, but only when the cost function of effort is sufficiently convex. In a similar framework, Moldovanu and Sela (2006) analyze whether it is better to organize one pooled contest, or a series of sub-contests whose winners compete against each other and whose losers are eliminated. The answer to this question now depends on the objective of the organizer (maximize expected total effort or maximize expected highest effort) and again on the curvature of the cost function of effort.[3]

In this paper we analyze the related but different case in which the organizer of a tournament has the option to allocate the available prize money over several parallel tournaments, after which participants have to choose which tournament to enter. While Moldovanu and Sela mentioned the interest of this issue already 10 years ago, we are not aware of any other study addressing this question.[4] An explanation for this might be that in their framework, where order statistics play an important role, incorporating participants' self-selection into the analysis is not innocuous.

In this paper we adopt the model of Tullock (1980), but allow the total prize money to be split (in unequal shares) over two tournaments. Azmat and Möller (2009) also adopt the framework of Tullock. They study the competition between organizers of tournaments. Organizers choose the prize structure to compete for homogeneous agents. In our model we allow agents to differ in ability and the total prize can differs between tournaments. Heterogeneous agents have to decide either to enter the tournament with the high prize or with the low prize. After learning who participates in which tournament, all agents decide how much effort to devote to winning the prize in their tournament.

Within our relatively simple framework we show that when agents select their tournament many different equilibria can arise. If the difference is prize money between the tournaments is sufficiently large, higher ability agents are more likely

[3]Szymanski (2003) provides a survey of different contest forms in sports, and gives many examples.

[4]In their concluding section, Moldovanu and Sela (2001) write that "[a]nother interesting extension would be the study of several parallel contests (with potentially different prize structures), such that agents can choose where to compete." And in the concluding section of Moldovanu and Sela (2006), they state that "[a]nother important avenue for future research is embedding the present analysis in a model of competition among contest designers. ... Since the contest architecture influences the expected payoffs of the participating agents, it is interesting to analyze which agents engage in which contests."

to sort into the tournament with the high prize than low-ability agents. However, if this difference is relatively small there may be multiple equilibria also including equilibria with reversed sorting by ability. Furthermore, we show that splitting tournaments does not increase total effort devoted by all agents compared to having only a single tournament. However, splitting the tournament may in some cases yield more effort from low-ability agents.

The outline of the paper is as follows. Section 2 provides the case where there is only a single tournament in which all agents participate. In Section 3 we discuss splitting the prize money over two tournaments. In Section 4 we allow a social planner to assign agents to tournaments and set the prizes. Section 5 concludes.

2 A simple tournament setting

Consider a setting with two types of agents. There are N_L low-ability agents having high constant marginal costs c_L of effort $e \geq 0$, and N_H high-ability agents with low marginal costs c_H of effort; $c_L > c_H > 0$. All agents participate in a tournament, where only the winner gets a prize M. Like Tullock (1980), we define the probability that agent i wins the prize by

$$p_i = \frac{e_i}{\sum_j e_j}$$

This success probability follows from the score function $s_i = \log(e_i) + \epsilon_i$, where ϵ_i follows an extreme type-I distribution, and the winner of the tournament is the agent with the highest score.

The expected utility of a risk-neutral agent equals

$$u_i = \frac{e_i}{\sum_j e_j} M - c_i e_i$$

Agents choose their effort to maximize expected utility. The first-order condition for agent i is

$$\frac{\partial u_i}{\partial e_i} = \frac{\sum_j e_j - e_i}{\left(\sum_j e_j\right)^2} M - c_i = 0$$

In equilibrium, all low-ability agents should devote the same effort e_L, and all high-ability agents devote effort e_H. The optimal effort for both groups is given by

$$e_L^* = \begin{cases} \frac{(N_L+N_H-1)(c_L+c_H N_H - c_L N_H)}{(N_L c_L + N_H c_H)^2} M & \text{if} \quad N_H \leq \frac{c_L}{c_L - c_H} \\ 0 & \text{if} \quad N_H > \frac{c_L}{c_L - c_H} \end{cases}$$

and

$$e_H^* = \begin{cases} \frac{(N_L+N_H-1)(c_H+c_L N_L-c_H N_L)}{(N_L c_L+N_H c_H)^2} M & \text{if} \quad N_H \le \frac{c_L}{c_L-c_H} \\ \frac{(N_H-1)c_H}{c_H^2 N_H^2} M & \text{if} \quad N_H > \frac{c_L}{c_L-c_H} \end{cases}$$

The condition $N_H \le \frac{c_L}{c_L-c_H}$ for positive effort of low-ability agents ensures that $c_L + c_H N_H - c_L N_H \ge 0$, so that effort both types of agents will never be negative. Note that this condition does not depend on the size of the prize M. Obviously, low-ability agents only participate in the tournament if the number of high-ability agents is limited. The upper bound for the number of high-ability agents depends on the relative difference in marginal costs of effort between low-ability and high-ability agents. Furthermore, if there is only one high-ability agent in the tournament, low-ability agents will always devote positive effort regardless of the relative cost difference.

The organizer of the tournament might be interested in the total effort devoted by all agents, which is given by

$$N_L e_L^* + N_H e_H^* = \begin{cases} \frac{N_L+N_H-1}{N_L c_L+N_H c_H} M & \text{if} \quad N_H \le \frac{c_L}{c_L-c_H} \\ \frac{N_H-1}{c_H N_H} M & \text{if} \quad N_H > \frac{c_L}{c_L-c_H} \end{cases}$$

Total effort thus linearly increases with the prize M. Furthermore, total effort is increasing in the number of low-ability agents N_L as long as $N_H \le \frac{c_L}{c_L-c_H}$, and otherwise total effort is unaffected by N_L. Increasing the number of high-ability agents N_H always increases total effort. This implies that adding additional agents to the tournament can never reduce the total effort devoted by all agents.

Using the expressions for total effort, we can easily derive expected utility. The expected utility of a high-ability agent is

$$u_H = \begin{cases} \frac{(c_H+c_L N_L-c_H N_L)^2}{(N_L c_L+N_H c_H)^2} M & \text{if} \quad N_H \le \frac{c_L}{c_L-c_H} \\ \frac{1}{N_H^2} M & \text{if} \quad N_H > \frac{c_L}{c_L-c_H} \end{cases}$$

and of a low-ability agent

$$u_L = \begin{cases} \frac{(c_L+c_H N_H-c_L N_H)^2}{(N_L c_L+N_H c_H)^2} M & \text{if} \quad N_H \le \frac{c_L}{c_L-c_H} \\ 0 & \text{if} \quad N_H > \frac{c_L}{c_L-c_H} \end{cases}$$

Of course, the expected utility of a high-ability agent is higher than the expected utility of a low-ability agent provided that there are multiple agents, i.e. $N_H + N_L > 1$.

3 Splitting the tournament

Next, we split the prize into two prizes αM and $(1-\alpha)M$ with $0.5 \leq \alpha \leq 1$. The prizes can be won in two different tournaments, and each agent is allowed to participate in only one of the two tournaments. The game is such that first agents choose whether they want to participate in the tournament with the high prize αM or the low prize $(1-\alpha)M$. After agents have chosen their tournament, they observe the tournament choices of all other agents, and determine their level of effort. Below, we provide a characterization of the equilibria for different values of α. When choosing the tournament we impose symmetry with respect to the own group. Agents with the same cost function of effort thus always have the same strategy.

3.1 All high-prize equilibrium

The first possible equilibrium we consider is the one where all agents (of both types) choose to participate in the tournament with the high prize αM. The consequence is that a deviating agent is the only participant in the low-prize tournament. With minimum effort this agent wins the prize, and (expected) utility of the deviating agent is therefore $(1-\alpha)M$.

Recall from the previous section that if a high-ability agent and a low-ability agent participate in the same tournament, then the high-ability agent has a higher expected utility. Therefore, if a low-ability agent does not deviate from participating in the high-prize tournament, the high-ability agent will not deviate either.

If all agents participate in the high-prize tournament, the expected utility of a low-ability agent is

$$u_L = \begin{cases} \frac{(c_L+c_H N_H - c_L N_H)^2}{(N_L c_L + N_H c_H)^2}\alpha M & \text{if} \quad N_H \leq \frac{c_L}{c_L - c_H} \\ 0 & \text{if} \quad N_H > \frac{c_L}{c_L - c_H} \end{cases}$$

Since this should exceed $(1-\alpha)M$, two conditions must be satisfied to ensure that all agents participate in the high-prize tournament. First, $N_H \leq \frac{c_L}{c_L - c_H}$ to ensure that low-ability agents have a positive expected utility in the high-prize tournament. And second, low-ability agents choose the high-prize tournament if $\frac{(c_L+c_H N_H - c_L N_H)^2}{(N_L c_L + N_H c_H)^2}\alpha M \geq (1-\alpha)M$. Rewriting gives the following conditions for the all high-prize equilibrium

$$\alpha \geq \frac{(N_L c_L + N_H c_H)^2}{(N_L c_L + N_H c_H)^2 + (c_L + c_H N_H - c_L N_H)^2} \qquad \text{with} \qquad N_H \leq \frac{c_L}{c_L - c_H}$$

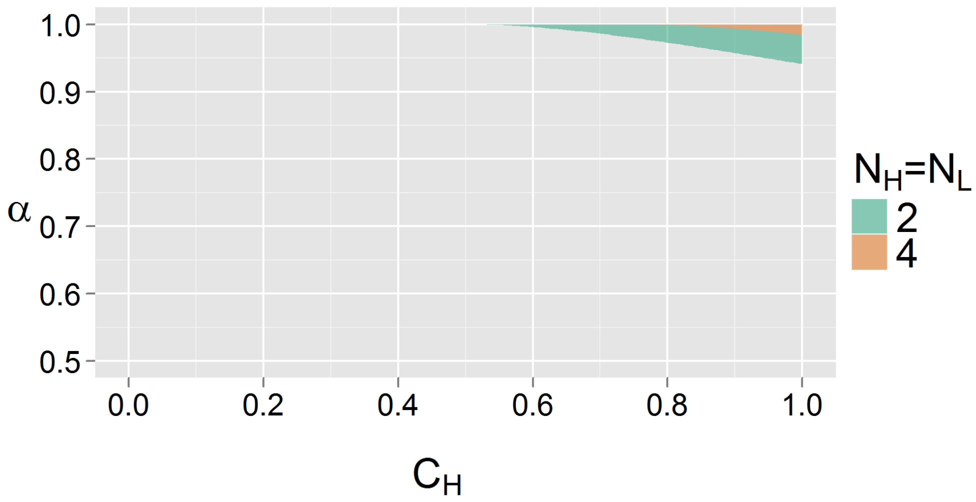

Figure 1: All-high-prize equilibrium

Note that both conditions do not depend on the total prize M.

If the number of high-ability agents is sufficiently small such that $N_H \leq \frac{c_L}{c_L - c_H}$, then the right-hand side of the first condition is less than or equal to 1. Furthermore, the right-hand side of the first condition is increasing in both N_H and N_L implying that if the number of agents increases, the fraction of the total prize going to the high-prize tournament should become higher to ensure that all agents choose this tournament. If the conditions above are satisfied, this equilibrium is unique.

It may be obvious that setting α such that all agents choose to participate in the high-prize tournament cannot increase total effort devoted by all agents. Recall from the previous section that total effort of all agents in a single tournament is linearly increasing in the tournament's prize. So if all agents participate in a tournament with a prize αM rather than M as would be in a single tournament, then all agents reduce their effort proportionally. The same holds for the expected utility of the agents, which is also a linear function of the prize money in the tournament.

To illustrate when this equilibrium exists we consider a numerical example. In this example we normalize the marginal costs of effort of the low-ability agents to one, i.e. $c_L = 1$. We consider two specific cases, first $N_H = N_L = 2$, and second $N_H = N_L = 4$. So one case with relatively few agents and one case with more agents. Figure 1 shows for which $0 < c_H < 1$ and $0.5 \leq \alpha \leq 1$ the all high-prize equilibrium arises. This equilibrium is only possible if α is relatively close to 1, which implies that almost all available prize money should be assigned to the high-prize tournament. Furthermore, the difference in marginal costs of effort between the high-ability and low-ability agents should not be too large, which is particularly true if the number of agents increases. In general the area at which in equilibrium all agents choose the high-prize tournament becomes smaller as the number of agents increases.

3.2 Perfect-sorting equilibrium

Next, we consider a second equilibrium in which the low-ability agents choose to participate in the tournament with the low prize $(1-\alpha)M$ and the high-ability agents participate in the tournament with the high prize αM. With this type of perfect sorting, the optimal effort of the high-ability agents becomes

$$e_H^* = \frac{(N_H-1)c_H}{c_H^2 N_H^2}\alpha M$$

and of the low-ability agents

$$e_L^* = \frac{(N_L-1)c_L}{c_L^2 N_L^2}(1-\alpha)M$$

There are two equilibrium conditions. First, high-ability agents should prefer the high-prize tournament over competing in the low-prize tournament with all the low-ability agents. And second, the low-ability agents should not want to deviate to the high-prize tournament competing with all the high-ability agents.

First, consider the high-ability agents. When participating in the tournament with the high prize, their expected utility is $\frac{1}{N_H^2}\alpha M$. Switching to the low-prize tournament gives an expected utility $\frac{(c_H+c_L N_L-c_H N_L)^2}{(c_H+c_L N_L)^2}(1-\alpha)M$. Expected utility from the high-prize tournament exceeds expected utility from the low-prize tournament if

$$\alpha \geq \frac{N_H^2\left(c_H+c_L N_L-c_H N_L\right)^2}{(c_H+c_L N_L)^2+N_H^2\left(c_H+c_L N_L-c_H N_L\right)^2}$$

This lower-bound restriction is strictly smaller than 1.

To determine the upper bound restriction on α, consider a low-ability agent. For this agent it should not be beneficial to switch to the high-prize tournament. In the low-prize tournament the expected utility is $\frac{1}{N_L^2}(1-\alpha)M$, and in the high-prize tournament $\frac{(c_L+c_H N_H-c_L N_H)^2}{(c_L+c_H N_H)^2}\alpha M$. Furthermore, switching can only be beneficial if $N_H \leq \frac{c_L}{c_L-c_H}$, otherwise the low-ability agent devotes no effort in the high-prize tournament. A low-ability agent does not want to switch from the low-prize tournament to the high-prize tournament if

$$\alpha \leq \frac{(c_L+c_H N_H)^2}{(c_L+c_H N_H)^2+N_L^2(c_L+c_H N_H-c_L N_H)^2} \qquad \text{with} \qquad N_H \leq \frac{c_L}{c_L-c_H}$$

If the second inequality is not satisfied, the upper-bound restriction simplifies to $\alpha \leq 1$.

Collecting the restrictions gives the conditions for the perfect-sorting equilibrium:

$$\frac{N_H^2\left(c_H + c_L N_L - c_H N_L\right)^2}{(c_H + c_L N_L)^2 + N_H^2\left(c_H + c_L N_L - c_H N_L\right)^2} \leq \alpha \leq \min\left(\frac{(c_L + c_H N_H)^2}{(c_L + c_H N_H)^2 + N_L^2(c_L + c_H N_H - c_L N_H)^2}, 1\right)$$

The inequalities show that both the lower bound and the upper bound increase as the number of high-ability agents N_H increases. In the first inequality, the upper bound decreases in the number of low-ability agents N_L, and this is also the case for the lower bound in both inequalities. If c_H becomes larger the lower bound decreases, as well as the upper bound. The intuition behind this is that if the marginal costs of effort of the high-ability increase, they become more similar to the low-ability agents, and, therefore, the prize money should be divided more equally among both tournaments to have perfect sorting as an equilibrium.

In the perfect-sorting equilibrium, the total effort of all agents is given by

$$N_L e_L^* + N_H e_H^* = \frac{N_L - 1}{c_L N_L}(1-\alpha)M + \frac{N_H - 1}{c_H N_H}\alpha M$$

It is interesting to compare this to the total effort in a single tournament. Recall that in the single tournament, we could distinguish two cases. First, if $N_H \leq \frac{c_L}{c_L - c_H}$, both the high-ability and low-ability agents devote positive effort, and total effort was given by $\frac{N_L + N_H - 1}{N_L c_L + N_H c_H} M$. To see how splitting the tournament in this case affects total effort, we should consider

$$\frac{N_L - 1}{c_L N_L}(1-\alpha)M + \frac{N_H - 1}{c_H N_H}\alpha M - \frac{N_L + N_H - 1}{N_L c_L + N_H c_H} M$$

All terms in this expression are linear in M. When ignoring M, this can be rewritten to

$$c_H N_H^2(c_H(N_L - 1) - c_L N_L) + \alpha c_L N_L^2(c_L(N_H - 1) - c_H N_H)$$

This is always negative. The first term is negative because high-ability agents have lower marginals costs of effort than low-ability agents and, therefore, $c_H(N_L - 1) < c_L N_L$. The second term is only positive if $c_L(N_H - 1) - c_H N_H > 0$, which implies $N_H > \frac{c_L}{c_L - c_H}$. However, if this is satisfied, low-ability agents do not devote any effort, and in that case we should evaluate

$$\frac{N_L - 1}{c_L N_L}(1-\alpha)M + \frac{N_H - 1}{c_H N_H}\alpha M - \frac{N_H - 1}{c_H N_H} M$$

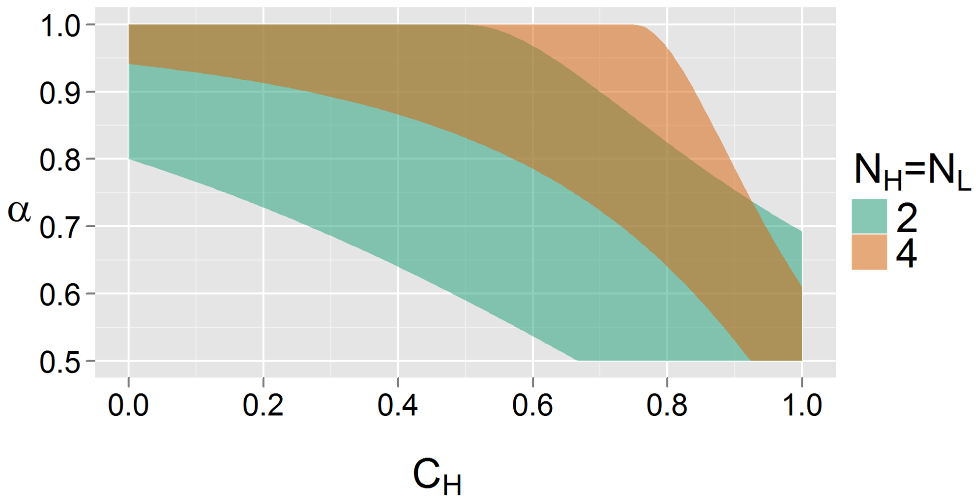

Figure 2: Perfect-sorting equilibrium

If we again ignore M, we can rewrite this expression to

$$\frac{N_L - 1}{c_L N_L} - \frac{N_H - 1}{c_H N_H}$$

The condition $N_H > \frac{c_L}{c_L - c_H}$ implies $\frac{(N_H - 1)}{c_H N_H} > \frac{1}{c_L} \left(= \frac{N_L}{c_L N_L}\right) > \frac{N_L - 1}{c_L N_L}$. The main conclusion is thus that if the value of α is such that there is a perfect-sorting equilibrium, total effort is lower than in case of a single tournament.

Let us return to the numerical example discussed in the previous subsection. Figure 2 shows the areas where perfect sorting is an equilibrium. The figure indicates that for perfect sorting to be an equilibrium it is either required that the difference in marginal costs of effort between the high-ability and low-ability agents is relatively large (c_L is low), or (if both groups have relatively similar marginal costs of effort) the smaller prize should not be too large (α too small). If the difference in marginal costs of effort is large and the low prize is also substantial, perfect sorting is not an equilibrium. In that case it becomes beneficial for the high-ability agents to enter the low-prize tournament, and there is a mixing equilibrium (which we will discuss below). Increasing both the number of high-ability and low-ability agents causes the lower bound to increase. The effect on the upper bound is not monotonic. Recall that the number of high-ability agents and the number of low-ability agents have an opposite effect on the direction in which both bounds move.

Non-uniqueness of the perfect-sorting equilibrium

Unlike the equilibrium where all sort into the high-prize tournament, the perfect-sorting equilibrium is not necessarily unique. When the number of high-ability agents is low reverse sorting might also be an equilibrium. In that case, all low-ability agents are in the high-prize tournament, and the high-ability agents are in the low-prize tournament. A high-ability agent prefers to participate in the low-prize tournament with only a small number of high-ability agents than to enter the high-prize tournament with more low-ability agents.

This reverse-sorting equilibrium can only arise if the following conditions are satisfied. First, it should not be beneficial for high-ability agents to deviate, which means,

$$\frac{1}{N_H^2}(1-\alpha)M \geq \frac{(c_H + c_L N_L - c_H N_L)^2}{(c_H + c_L N_L)^2}\alpha M$$

Recall that if there is only a single high-ability agent in a tournament, low-ability agents will devote positive effort. The upper-bound condition for the reverse-sorting equilibrium is

$$\alpha \leq \frac{(c_H + c_L N_L)^2}{(c_H + c_L N_L)^2 + N_H^2 \left(c_H + c_L N_L - c_H N_L\right)^2}$$

From this condition it can be seen that if the number of high-ability agents N_H is large, the upper bound is below 0.5 implying that reverse sorting is not an equilibrium. However, if N_H is small, then the upper bound is above 0.5, which can be verified by considering the case $N_H = 1$. Furthermore, the upper bound increases in the number of low-ability agents N_L.

Second, also for a low-ability agent it should not be beneficial to deviate, which implies

$$\frac{1}{N_L^2}\alpha M \geq \frac{(c_L + c_H N_H - c_L N_H)^2}{(c_L + c_H N_H)^2}(1-\alpha)M$$

The lower bound is thus given by

$$\alpha \geq \frac{N_L^2(c_L + c_H N_H - c_L N_H)^2}{(c_L + c_H N_H)^2 + N_L^2(c_L + c_H N_H - c_L N_H)^2}$$

And, of course, $N_H \leq \frac{c_L}{c_L - c_H}$, otherwise the deviating low-ability agent will not devote any effort when entering the low-prize tournament with all high-ability agents. For the lower-bound condition to exceed 0.5 the number of low-ability agents should be sufficiently large, i.e. $N_L > \frac{c_L + c_H N_H}{c_L + c_H N_H - c_L N_H}$.

Like in the perfect-sorting equilibrium, we can also show for the reverse-sorting equilibrium that total effort of all agents is always lower than in case of a single tournament. The proof is the same as the proof in Subsection 3.2.

Let us again consider the numerical example discussed earlier. Figure 3 shows for which combinations of c_H and α the reverse-sorting equilibrium exists. It is clear that c_H should be relatively close to 1, implying that the marginal costs of effort of the high-ability agents should be relatively close to the marginal costs of effort

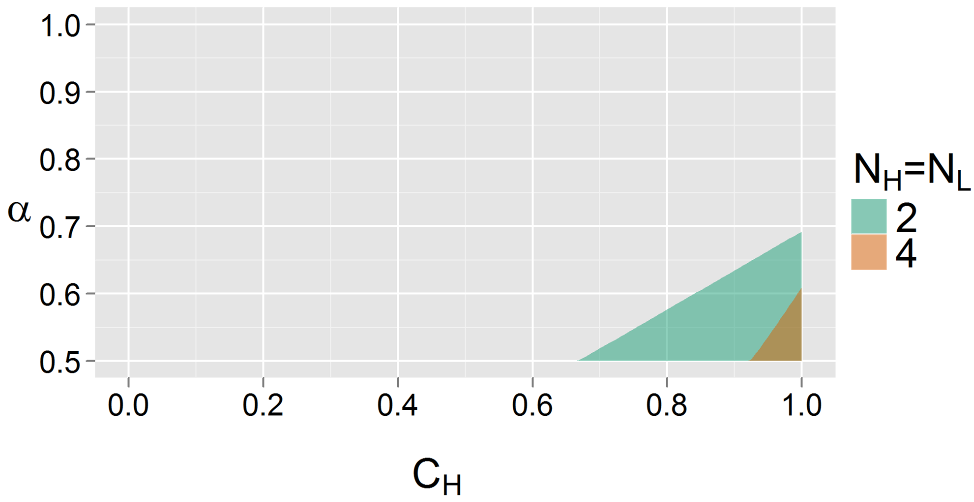

Figure 3: Reverse-sorting equilibrium

of the low-ability agents. If the difference becomes too large, reverse sorting is no longer an equilibrium. Furthermore, a substantial part of the total prize should be assigned to the low-prize tournament. As already mentioned above the area for which reverse sorting is an equilibrium shrinks rapidly if the number of (high-ability) agents increases. By comparing the area in which reverse sorting is an equilibrium with the area in which perfect sorting is an equilibrium (displayed in Figure 2), one can see that these areas overlap. This means that both equilibria are not necessarily unique.

3.3 Mixed-strategy equilibria

The numerical example discussed above shows that for some parameter values, there are multiple pure-strategy equilibria. Also for some parameter values there are no pure-strategy equilibria. In this subsection we consider mixed-strategy equilibria. There are a number of mixed-strategy equilibria, which we discuss below. It might be that both types of agents follow a mixed strategy or that one of the types has a pure strategy while the other type has a mixing strategy.

First, consider the possible equilibrium in which both the high-ability agents and the low-ability agents follow a mixed strategy. If α equals 0.5, the total prize is divided equally over both tournaments. So both tournaments are the same, and agents are indifferent in which tournament they participate. A possible equilibrium is that both a high-ability agent and a low-ability agent choose to enter each of the two tournaments with probability 0.5. Because both tournaments are ex-ante the same, none of the agents can improve their outcome by deviating from this mixing strategy.

There is not only a mixed-strategy equilibrium for α being exactly 0.5, but often also for higher values of α. Determining for a given value of the parameters the

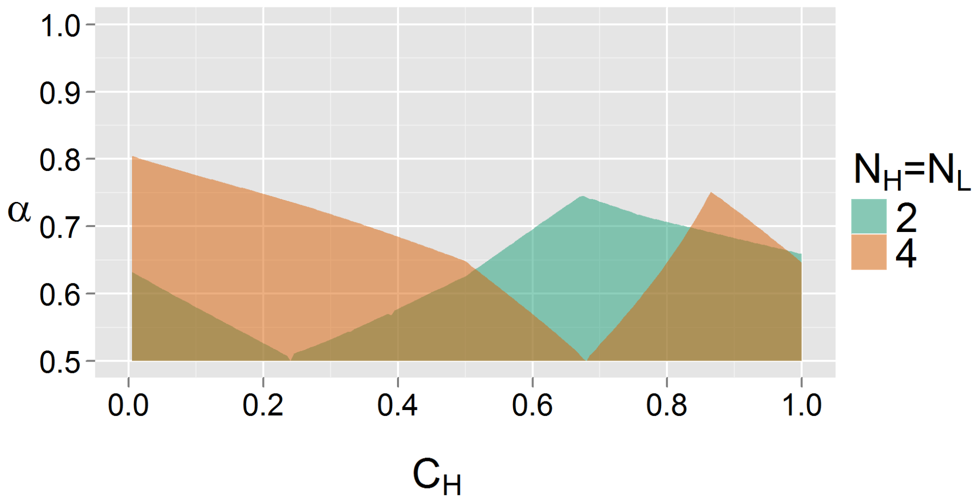

Figure 4: Mixed-strategy equilibrium – where both types mix

highest value for α for which it is an equilibrium that both agents have a mixed strategy is more complicated. Therefore, we use the numerical example discussed before and show in Figure 4 for which parameter values it is an equilibrium that both high and low-ability agents mix. The figure shows that for low values of c_H a mixed-strategy equilibrium is possible for higher values of α if there are more agents. For higher values of c_H the figure gets slightly more complicated to interpret. As we will see below, this has to do with other existing mixed-strategy equilibria. In particular, whether or not the low-ability agents may choose a pure strategy to enter the low-prize tournament. Furthermore, it should be noted that this mixed-strategy equilibrium partly overlaps with the perfect-sorting equilibrium and the reverse-sorting equilibrium. The intuition is that for values for c_H relatively close to one, α close to 0.5, and two high-ability and two low-ability agents it is an equilibrium if there are (in expectation) in each tournaments two agents. In that case no agent would want to change because it would imply competing against two agents rather than one.

In a mixed-strategy equilibrium it is difficult to evaluate the (expected) total effort devoted by all agents. Agents only decide about the effort after they learn which agents compete in which tournament. Since all agents follow a mixed strategy, the composition of the tournaments is stochastic and so is total effort. We will show in Section 4 that total effort will always be lower in mixed-strategy equilibria than when only having a single tournament.

From the different figures shown so far, it is clear that there are parameter values for which we have not yet shown any equilibrium. There are two remaining possible mixed-strategy equilibria. The first is the equilibrium in which all high-ability agents decide to participate in the high-prize tournament, while the low-ability agents mix between both tournaments. Recall from Subsection 3.1 that the all high-prize equilibrium is unique. So a first condition for having a mixed-strategy equilibrium in which only low-ability agents mix is that $\alpha < \frac{(N_L c_L + N_H c_H)^2}{(N_L c_L + N_H c_H)^2 + (c_L + c_H N_H - c_L N_H)^2}$ if $N_H < \frac{c_L}{c_L - c_H}$. Furthermore, a low-ability agent should prefer to enter the high-prize tournament if all other low-ability agents enter the low-prize tournament. From Subsection 3.2 we know that this implies $N_H < \frac{c_L}{c_L - c_H}$ and $\alpha > \frac{(c_L + c_H N_H)^2}{(c_L + c_H N_H)^2 + N_L^2 (c_L + c_H N_H - c_L N_H)^2}$. This second mixed-strategy equilibrium is thus possible if $N_H < \frac{c_L}{c_L - c_H}$ and

$$\frac{(N_L c_L + N_H c_H)^2}{(N_L c_L + N_H c_H)^2 + (c_L + c_H N_H - c_L N_H)^2} < \alpha < \frac{(c_L + c_H N_H)^2}{(c_L + c_H N_H)^2 + N_L^2 (c_L + c_H N_H - c_L N_H)^2}$$

This is exactly the area between the all high-prize equilibrium displayed in Figure 1 and the perfect-sorting equilibrium shown in Figure 2.

The final mixed-strategy equilibrium is one where all low-ability agents sort into the low-prize tournament and the high-ability agents mix over both tournaments. Obviously, this equilibrium requires that a high-ability agent should prefer the low-prize tournament if all other high-ability agents enter the high-prize tournament. The upper bound for α for this mixed-strategy equilibrium is thus the lower bound of the perfect-sorting equilibrium, and is given by

$$\alpha < \frac{N_H^2 (c_H + c_L N_L - c_H N_L)^2}{(c_H + c_L N_L)^2 + N_H^2 (c_H + c_L N_L - c_H N_L)^2}$$

Determining the lower bound of this mixed-strategy equilibrium is more complicated and should again be done numerically.

In Figure 5 we again consider the example and show for which values of c_L and α this equilibrium arises. For a setting with only few agents, the value of c_H should be sufficiently small and furthermore α should not be too high. If this is not the case, there will be a perfect selection equilibrium. Furthermore, if both c_H and α are very low, not only the high-ability agents but also the low-ability agents follow a mixed strategy. As can be seen from the figure the area indicating the equilibrium shift substantially if there are more agents in the tournament. The latter implies that for more parameter values the equilibrium in unique in which both types of agents use a mixed strategy. Indeed, if there are many agents of both types, the high-ability agents follow a mixed strategy and enter both the low-prize and the high-prize tournament. If the number of high-ability agents is sufficiently large, then even low-ability agents in the low-prize tournament will not devote any effort.

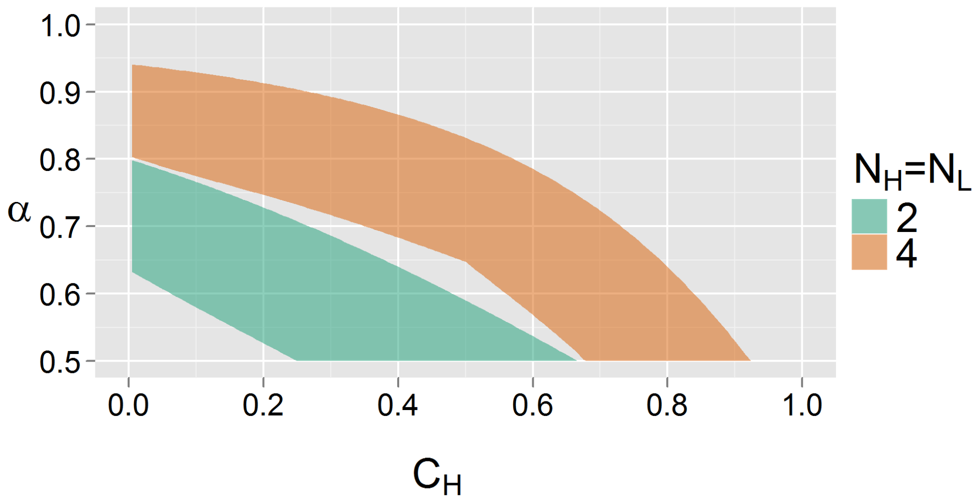

Figure 5: Mixed-strategy equilibrium – where only high-ability types mix

In the limit only the behavior of the high-ability agents is relevant and they mix over both tournaments.

4 Social planner

So far, we have mainly considered the behavior of agents. We have largely ignored the organizer of the tournament, although we implicitly assumed that the organizer's objective is to optimize total effort of all agents. Below we assume that there is a social planner, who can not only decide about the value of α, but can also assign the agents to the tournaments. The social planner might not only be interested in maximizing total effort, but we also consider a second objective: maximizing the minimum effort level. For illustration we start by considering the case where $N_H = N_L = 2$. Obviously, the social planner should make sure that all agents are actually competing, implying that each agent should have at least one competitor. An agent who does not have any competitor earns a prize without providing any effort. Given both objectives of the social planner, this can never be optimal. This leaves us with four possible assignments:

- All four agents are assigned to a single tournament in which the full prize money (M) can be won (the pooled tournament).
- One high-ability agent and one low-ability agent are assigned to the high-prize tournament and one high-ability agent and one low-ability agent are assigned to the low-prize tournament (two mixed tournaments).
- The two high-ability agents are assigned to the high-prize tournament and the two low-prize agents are assigned to the low-prize tournament (perfect sorting).

- The two high-ability agents are assigned to the low-prize tournament and the two low-prize agents are assigned to the high-prize tournament (reverse sorting).

We use the results derived earlier to report in Table 1 the maximum total effort and the maximum minimum effort that can be achieved under each of these assignments. Above the equation-signs we report the value of α for which these outcomes are attainable.

It is easy to check that the maximum total effort in the pooled tournament is always as least as large as the maximum total effort in case of a split tournament. Consider in particular the case of perfect sorting. The best the social planner can do in that case is to set $\alpha = 1$. But if all the prize money goes to the high-prize tournament in which the high-ability agents participate, it is better to re-assign the low-ability agents also to the high-prize tournament. As we have shown in Section 2 increasing the number of agents in a tournament will never decrease the total effort in this tournament.

It is also easy to check that minimum effort under reverse sorting exceeds minimum effort under perfect sorting and under mixed sorting.[5] Minimum effort under reverse sorting is, however, not necessarily larger than minimum effort in the pooled tournament. This is only true if $c_H < \frac{4}{5}c_L$: the marginal cost of effort of the low-ability agents has to be sufficiently above the marginal cost of effort of the high-ability agents. Otherwise, it is better not to split the prize money and to run the pooled tournament.[6] Whether the pooled tournament induces higher or lower minimum effort than the mixed tournaments or the perfect-sorting tournaments, also depends on the values of c_L and c_H. Mixed tournaments dominate the pooled tournament as long as $c_H < \frac{3}{5}c_L$. Perfect-sorting tournaments dominate the pooled tournament if $10c_Hc_L - c_H^2 < 7c_L^2$, which holds if c_H is sufficiently small compared to c_L.

We can generalize the results regarding maximum total effort to other values of N_H and N_L. Once the social planner has divided all agents over both tournament, the optimal strategy would be to assign the full prize to one of the tournaments, i.e. the tournament with the highest "weight", because effort within each tournament is linear in the prize. If the full prize has been assigned to a single tournament, it is also optimal to assign all agents to this tournament, because more competition can never reduce total effort. This argument stresses that splitting a tournament can

[5] In both cases this holds because $c_L > c_H$.

[6] Note, however, that this situation will never occur when agents self-select into tournaments, since reverse sorting is only an equilibrium if the costs of high-ability and low-ability agents are sufficiently equal.

Table 1: Outcomes of different assignments under social planner

Players	Prize	Maximum total effort	Maximum minimum effort
$\{2H+2L$	$M \}$	$\frac{3}{2c_L+2c_H}M$ if $c_L \leq 2c_H$ $\frac{1}{2c_H}M$ if $c_L > 2c_H$	$\frac{3(2c_H-c_L)}{(2c_L+2c_H)^2}M$ if $c_L \leq 2c_H$ 0 if $c_L > 2c_H$
$1H+1L$ $1H+1L$	αM $(1-\alpha)M$	$\frac{1}{c_L+c_H}\alpha M + \frac{1}{c_L+c_H}(1-\alpha)M = \frac{1}{c_L+c_H}M$	$\frac{c_H}{(c_L+c_H)^2}(1-\alpha)M \overset{\alpha=0.5}{=} \frac{c_H}{4(c_L+c_H)^2}M$
$2H$ $2L$	αM $(1-\alpha)M$	$\frac{1}{2c_H}\alpha M + \frac{1}{2c_L}(1-\alpha)M \overset{\alpha=1}{=} \frac{1}{2c_H}M$	$\frac{1}{4c_L}(1-\alpha)M \overset{\alpha=0.5}{=} \frac{1}{8c_L}M$
$2H$ $2L$	$(1-\alpha)M$ αM	$\frac{1}{2c_L}\alpha M + \frac{1}{2c_H}(1-\alpha)M \overset{\alpha=0.5}{=} (\frac{1}{4c_L} + \frac{1}{4c_H})M$	$\min(\frac{1}{4c_L}\alpha M, \frac{1}{4c_H}(1-\alpha)M) \overset{\alpha=\frac{c_L}{c_H+c_L}}{=} \frac{1}{4(c_L+c_H)}M$

never increase total effort of all agents and can thus never be the optimal strategy of a principal or social planner who has the main objective to optimize total effort. A principal or social planner who divided the prize over multiple tournaments thus puts in their objective function some weight on the effort of the low-ability agents.

5 Conclusions

To examine how heterogeneous agents choose among two tournaments with different prizes, we analyzed a model in which agents differ in their constant marginal cost of effort and in which an agent's probability to win a tournament is equal to the agent's effort relative to the sum of effort of all agents in that tournament. We characterize different equilibria and show how these are related to parameter values, especially how the prize money is divided across tournaments, and the difference in marginal costs between high-ability and low-ability agents.

A pooling equilibrium in which all agents choose for the high-prize tournament arises if a large share of the prize money goes to the high-prize tournament and if high-ability and low-ability agents are fairly similar in terms of their marginal costs of effort. A perfect sorting equilibrium can arise for less extreme values of these parameters. We also show, however, that this equilibrium is not unique. For some parameter values, both perfect sorting and reverse sorting are equilibria. We also characterize three types of mixed strategy equilibria.

A common finding for all equilibria is that total effort will never exceed the total effort that is provided when all prize money goes to a single pooled tournament. Hence, a principal who wants to maximize total effort should never split a tournament in different smaller tournaments. This result is in line with the conclusions in Moldovanu and Sela (2001) who find that splitting one large prize into various smaller prizes within the same tournament will not increase total effort unless agents' cost functions are very convex. This suggests that our conclusion depends on our assumption of constant marginal costs. We have not analyzed this case because allowing for increasing marginal costs turns our model intractable.

We also find that splitting a tournament in different smaller tournaments may have a beneficial impact on the effort level of low-ability agents. This is particularly the case if low-ability and high-ability agents are sufficiently different. A principal or social planner who cares about the effort of low-ability agents may thus split the prize over multiple tournaments.

In a related paper, we report about a field experiment in which first years economics students at the University of Amsterdam had to choose between three tournaments with different prizes (see Leuven et al., 2008). In one tournament the prize was 5000 euros, in another tournament the prize was 3000 euros, and in one tournament the prize was 1000 euros. Within each tournament the best performing student on the final exam of a standard introductory microeconomics course would win the prize. If we use high school math grades as measure of ability, the observed sorting pattern is consistent with students playing mixed strategies. Some high-ability students chose to enter the low-prize tournament and some low-ability students entered the high-prize tournament. However, on average, more able students are more likely to enter the high-prize tournament.

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